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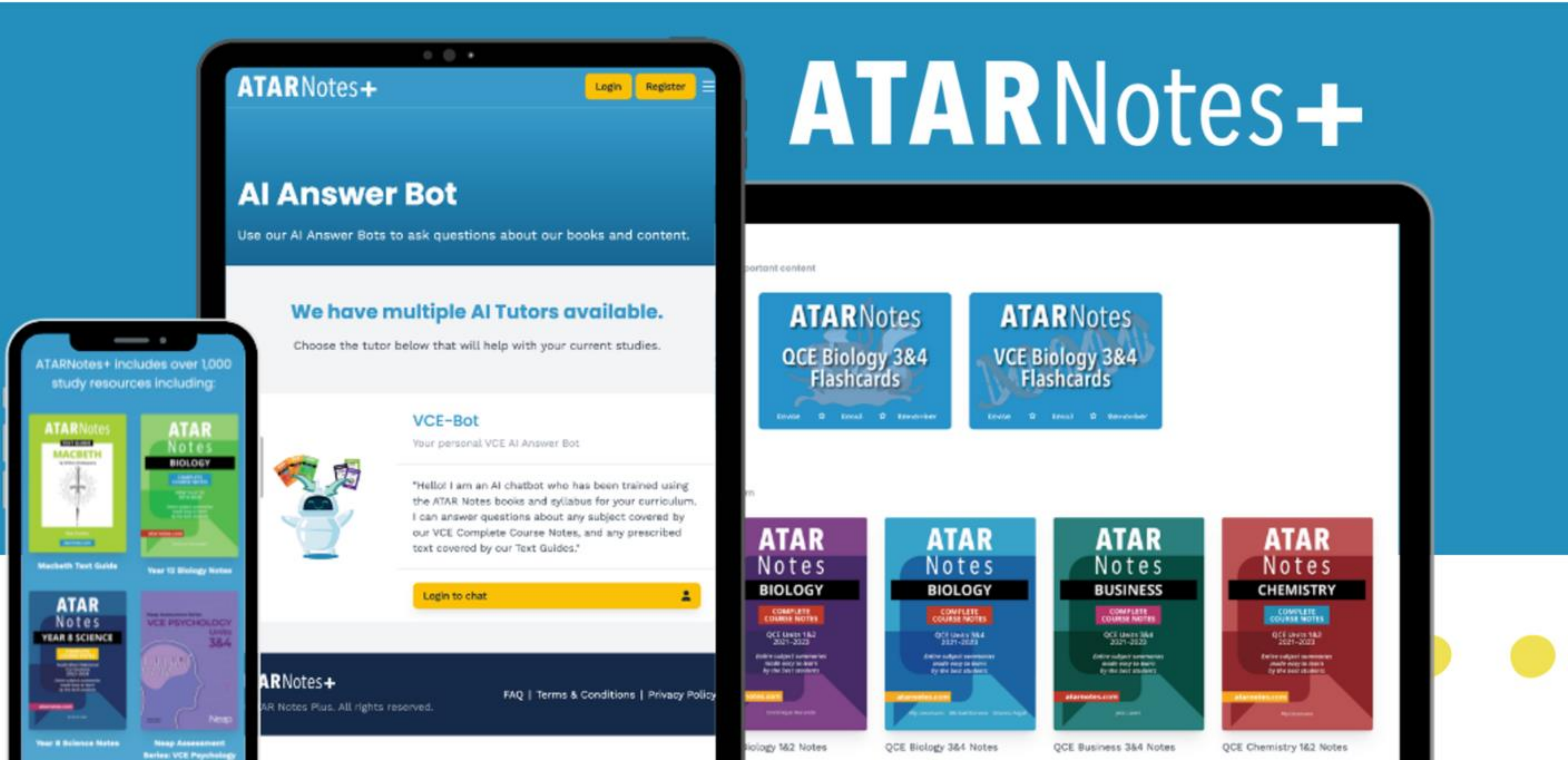


ATARNotes

Maths Methods 1/2

ATARNotes January Lecture Series

Presented by:
Kaif Qais

Overview

THE GAME PLAN

- Year 11 forms a strong basis for Year 12 – the goal is to provide a framework for you to build upon over the course of the first semester
- Content and practice questions
- Ask questions in the chat!

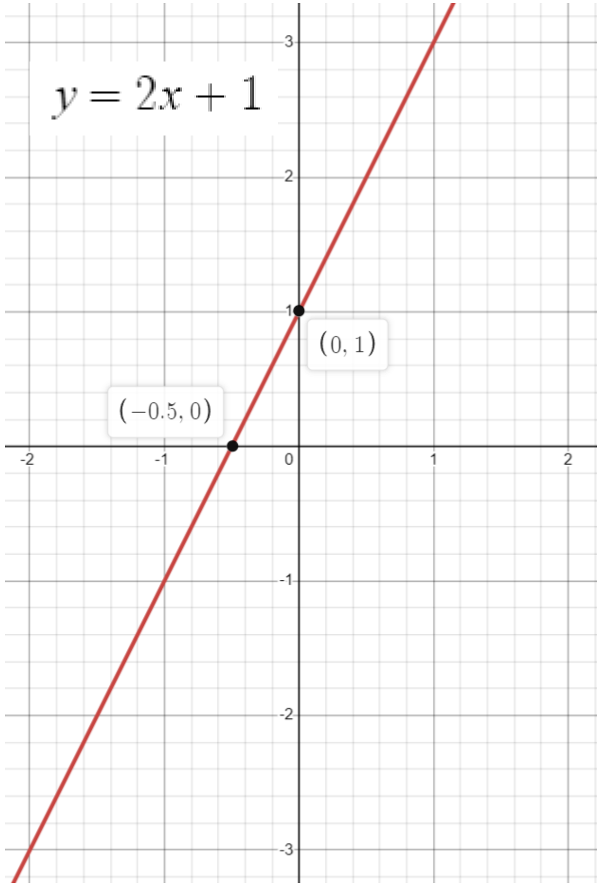
Topics Covered:

- Linear Geometry
  - Distance + Gradient
- Quadratics
  - Solving and factorising
  - Quadratic formula + Discriminant
- Transformations (DRT)
- Inverse functions
  - One to one functions
  - Properties
- Probability
  - Visualisation

Linear Geometry

A linear relation is a polynomial of degree one

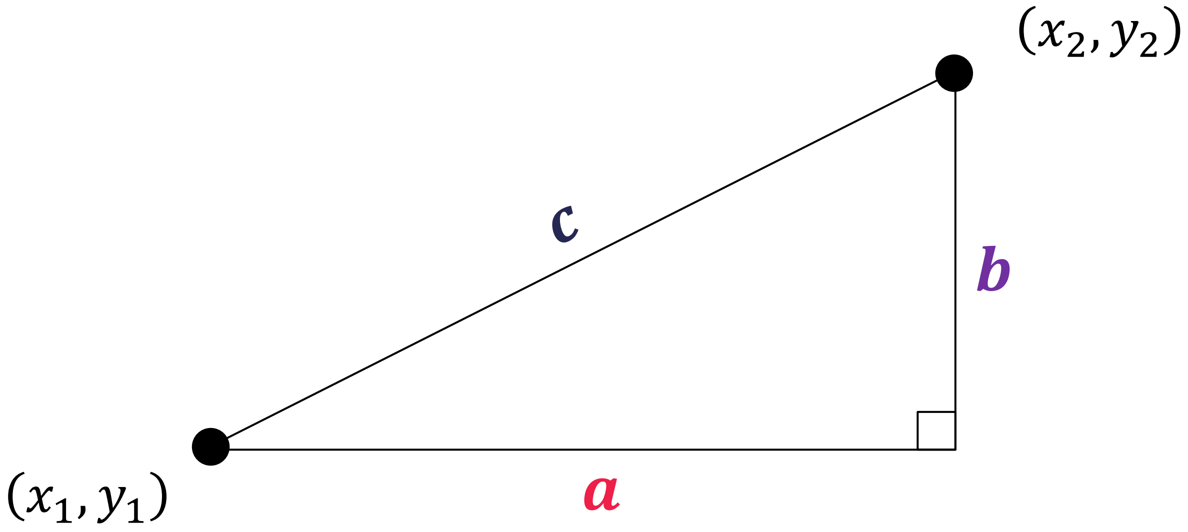
$$y = mx + c$$



- c is the y-intercept
- m is the gradient, or slope

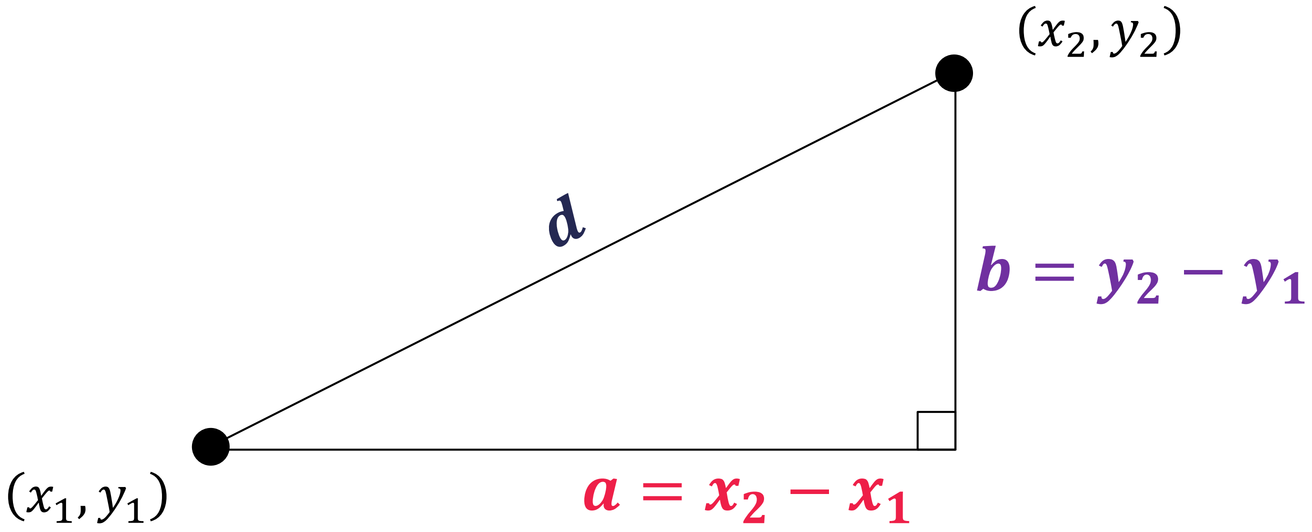
DISTANCE BETWEEN TWO POINTS

How could we find the distance between two points?



$$c^2 = a^2 + b^2$$

How could we find the distance between two points?



$$\boldsymbol{d} = \sqrt{(\boldsymbol{x_2 - x_1})^2 + (\boldsymbol{y_2 - y_1})^2}$$

DISTANCE BETWEEN TWO POINTS

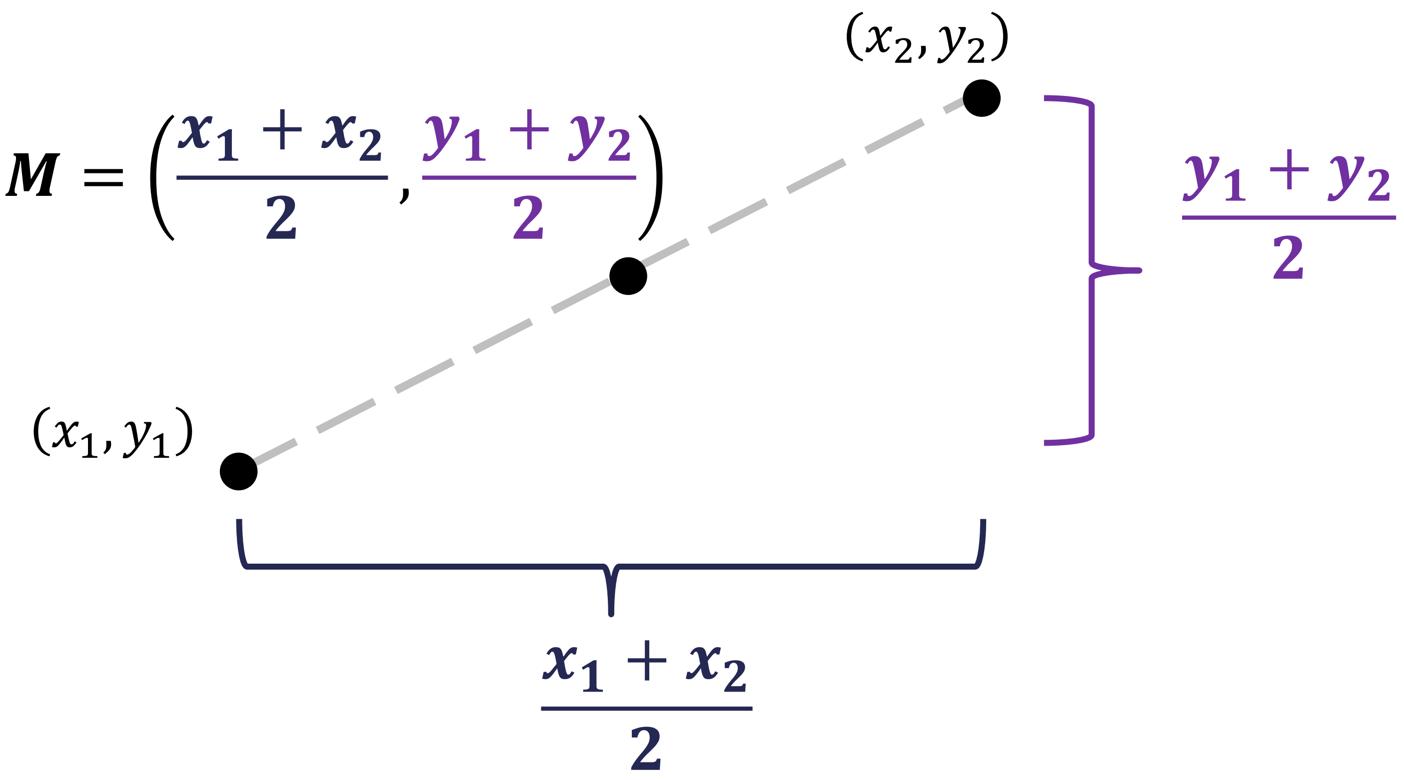
Some notes on the distance formula:

- **This formula isn't on the formula sheet**, so you will need to commit it to memory.
  - A good way to do this is to remember of the derivation of the formula with Pythagoras' theorem, like we just saw.

- When you're substituting in values, **it doesn't matter which point is (x_1, y_1) and which is (x_2, y_2)**.
  - This is because the formula calculates the difference between x-values and the difference between y-values.

How could we find the midpoint between two points?

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



Linear Geometry

The gradient of a line is referred to as $\frac{rise}{run}$.

$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Find the gradient of the line between the points (3, 2) and (1, 4).

$$m = \frac{y_2 - 2}{x_2 - 3}$$

You can use this to find the gradient between any two points on a graph (and hence the equation for the line)

$$m = \frac{4 - 2}{1 - 3}$$

$$m = \frac{2}{-2} = \mathbf{-1}$$

Linear Geometry

The gradient of a line is referred to as $\frac{rise}{run}$.

$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Find the gradient of the line between the points (3, 2) and (1, 4).

What if we choose the points in a different order

$$m = \frac{y_2 - 4}{x_2 - 1}$$

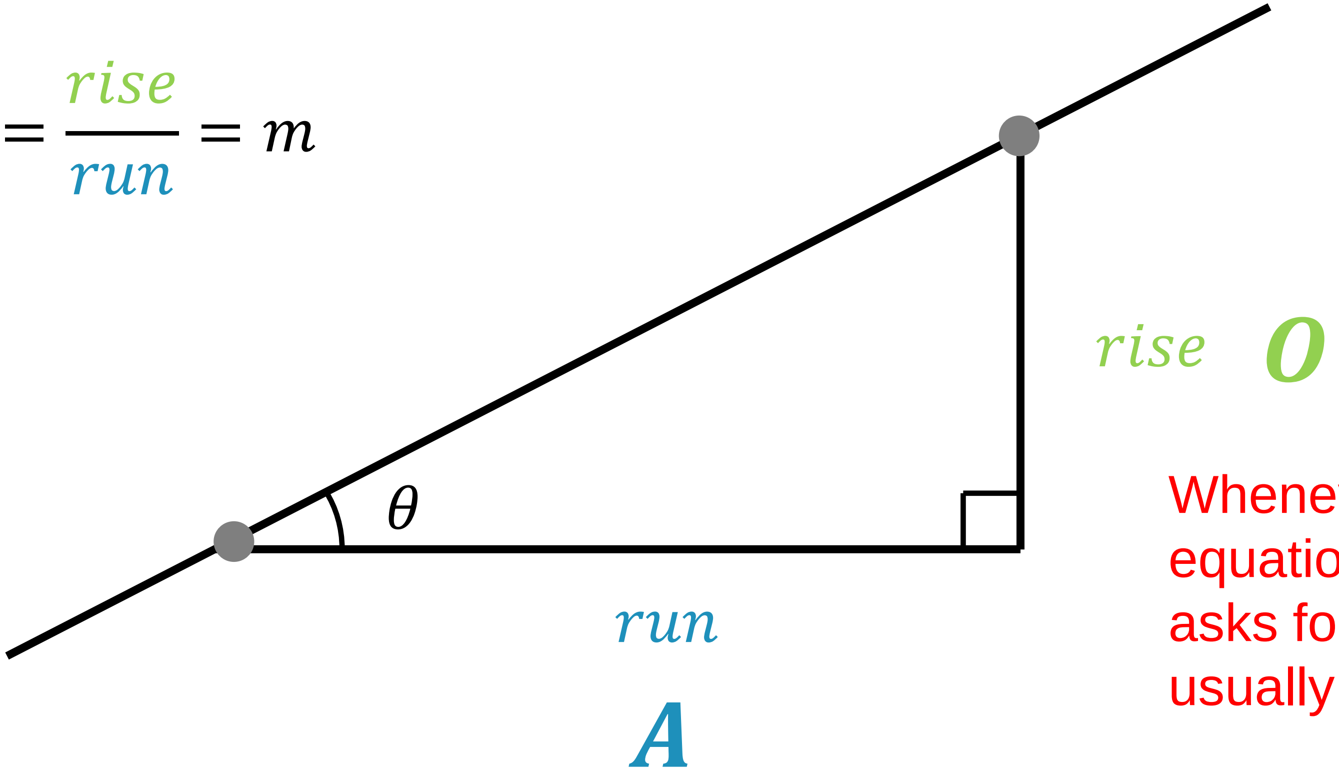
$$m = \frac{2 - 4}{3 - 1}$$

$$m = \frac{-2}{2} = \mathbf{-1}$$

GRADIENT

$$gradient = m = \frac{rise}{run}$$

$$\tan(\theta) = \frac{O}{A} = \frac{rise}{run} = m$$



Whenever a linear equation question asks for an angle – it usually involves this!

Linear Geometry

Q

A line passes through the points (1,2) and (-3,6). Find the equation of the line.

Formally, a polynomial is an equation in the form

$$y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x^1 + a_0$$

…where n is a positive, whole number.

$n = 1: y = mx + c$

$n = 2: y = ax^2 + bx + c$

n is referred to as the degree of the polynomial.

Think of it instead as an expression where x is raised to whole number powers!

Expanded form

- The expanded form of polynomials looks like:
  - Quadratics: $y = ax^2 + bx + c$
  - Cubics: $y = ax^3 + bx^2 + cx + d$
  - Quartics: $y = ax^4 + bx^3 + cx^2 + dx + e$
- The only information about the polynomial that it tells us is **the y-intercept**, given by the constant term.
  - (the y-intercept will occur when $x = 0$)
- To solve the quadratic when $y = 0$, we must either
  - Factor the quadratic (converting the polynomial to factorised form)
  - Complete the square (converting the polynomial to stationary point form)
  - Use the quadratic formula

Factorised form

- The factorised form of polynomials looks like:
  - Quadratics: $y = a(x+b)(x+c)$
  - Cubics: $y = a(x+b)(x+c)(x+d)$
  - Quartics: $y = a(x+b)(x+c)(x+d)(x+e)$
- Not all polynomials can be expressed in this form.
- If a polynomial can be expressed in factorised form, these linear factors will tell us **where the x-intercepts will be**.
  - (x-intercepts will occur when $y = 0$ so we can find them using the null factor law)

Stationary point form

- The stationary point form of polynomials looks like:
  - Quadratics: $y = a(x-b)^2 + c$
  - Cubics: $y = a(x-b)^3 + c$
  - Quartics: $y = a(x-b)^4 + c$

Be careful of the signs!

- Not all polynomials can be expressed in this form.
- If a polynomial can be expressed in this form, the polynomial will **only have one stationary point**, which will be **at** $(\boldsymbol{b}, \boldsymbol{c})$.

FACTORING A QUADRATIC

Solve $2x^2 + 9x + 9 = 0$ for x.

Often, there will be no a term; no number in front of the x^2 term, making it easier to factorise.

To convert a quadratic in expanded form ($y = ax^2 + bx + c$) to turning point form, we can **complete the square**.

The process for completing the square

$$y = x^2 + bx + c$$

$$= \underbrace{x^2 + bx + \frac{b^2}{4}} - \frac{b^2}{4} + c$$

$$= \overbrace{\left(x + \frac{b}{2}\right)^2} - \frac{b^2}{4} + c$$

Halve and square b. Add this term to make a perfect square. Then subtract this term, so the expression is overall the same as before.

This expression is now a perfect square

Q

$$y = x^2 - 6x - 1$$

Express the following quadratic in the form $y = a(x+b)^2 + c$.

Q

$$y = 3x^2 - 6x - 1$$

Express the following quadratic in the form $y = a(x + b)^2 + c$.

QUADRATIC FORMULA

- **Solutions to a quadratic equation** (something of the form $\boldsymbol{ax^2 + bx + c = 0}$) are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The quadratic formula comes from completing the square on the equation $ax^2 + bx + c = 0$, and then rearranging to solve for x.

Derivation of quadratic formula:

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 + \left(\frac{-b^2}{4a^2} + \frac{4ac}{4a^2}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Quadratic Formula (continued)

- E.g. Using the quadratic formula, find the solution(s) for $f: R \to R, f(x) = 2x^2 - 4x - 3$

Q

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-3)}}{4}$$

$$x = \frac{4 \pm \sqrt{16 - 4(2)(-3)}}{4}$$

$$x = \frac{4 \pm \sqrt{16 - 8(-3)}}{4}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{4 \pm \sqrt{40}}{4}$$

Key tip: The best way to avoid making mistakes on non-calc tests is writing out **EVERY** step.

The Quadratic Formula (continued)

- E.g. Using the quadratic formula, find the solution(s) for $f: R \to R, f(x) = 2x^2 - 4x - 3$

Q

$$x = \frac{4 \pm \sqrt{4 \times 10}}{4}$$

$$x = \frac{4 \pm 2\sqrt{10}}{4}$$

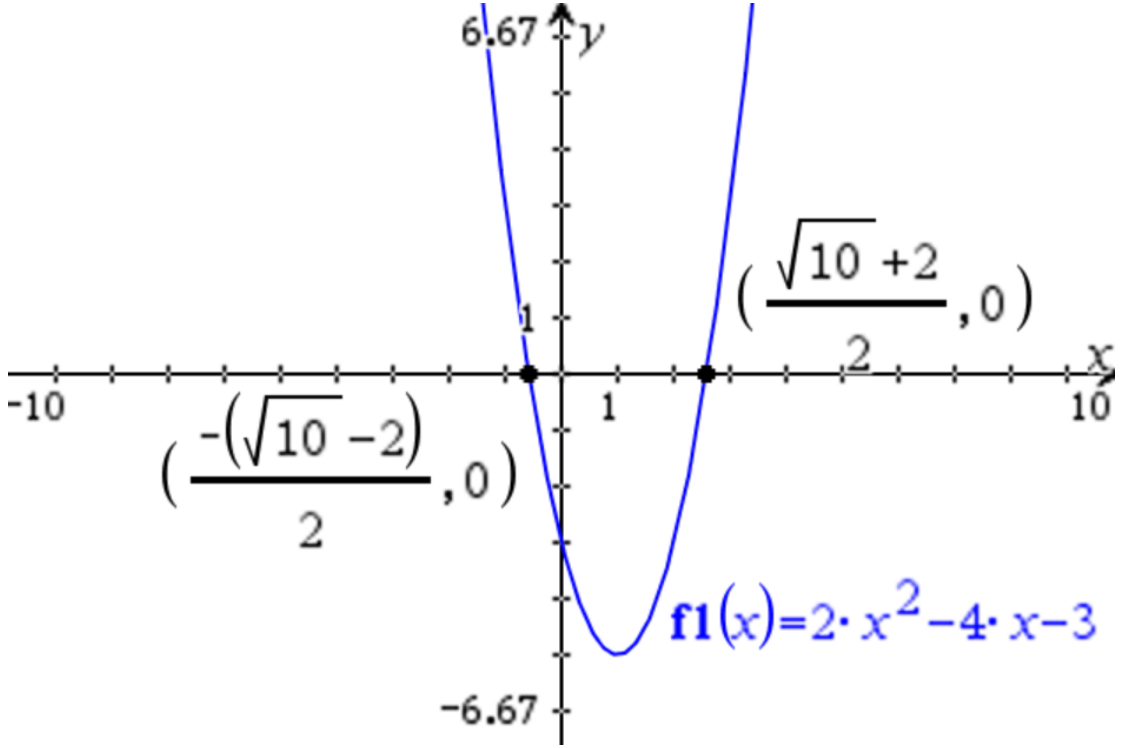
$$x = \frac{2 \pm \sqrt{10}}{2}$$

$$\therefore \boldsymbol{x = \frac{2 - \sqrt{10}}{2} \; or \; x = \frac{2 + \sqrt{10}}{2}}$$

The Quadratic Formula (continued)

- E.g. Using the quadratic formula, find the solution(s) for $f: R \to R, f(x) = 2x^2 - 4x - 3$

Q



SKETCHING TIPS

- Determine what form the quadratic is in; what info does it give you?
- Use algebraic techniques to derive the rest of the info
  - x & y intercepts
  - Stationary point
  - Axis of symmetry

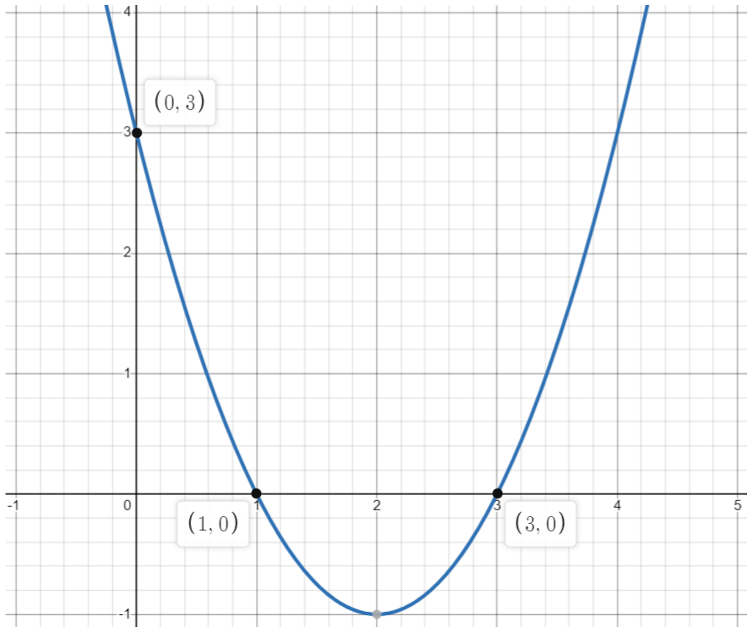
$y\ intercept = 3$

$x\ intercept$: $when\ y = 0,$
$0 = x^2 - 4x + 3$
$= (x-3)(x-1)$
$\therefore x = 3\ or\ x = 1$

$turning\ point$:
$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$$
$when\ x = 2,$
$y = 2^2 - 4(2) + 3 = -1$
$\therefore turning\ point\ at\ (2, -1)$

$axis\ of\ symmetry$: $x = 2$

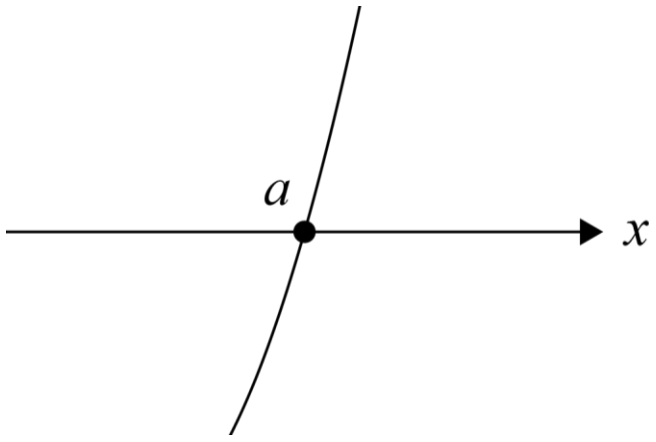
$y = x^2 - 4x + 3$



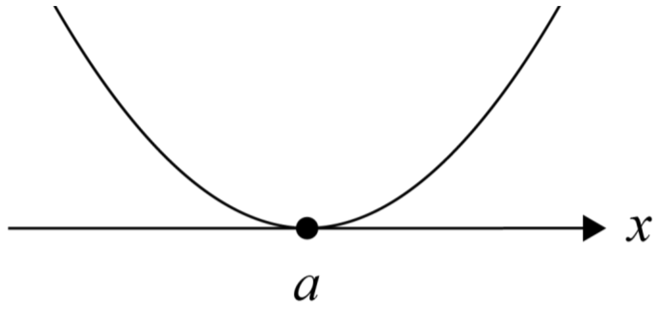
Factorised form

- If a factor is raised to some power, the nature of the x-intercept will change.

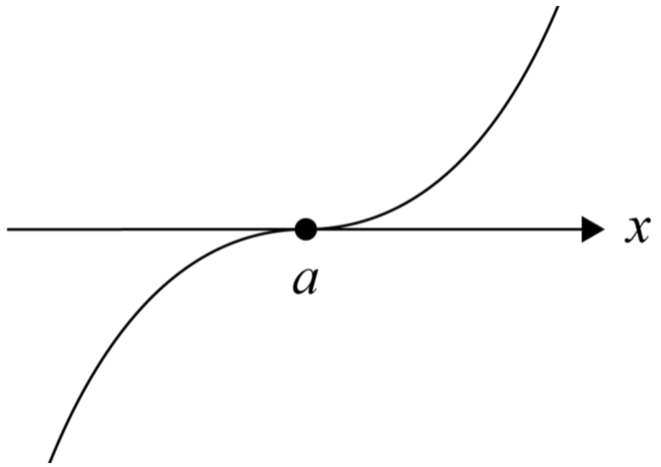
Factor: $(x - a)$



Factor: $(x - a)^2$



Factor: $(x - a)^3$



- Looking at the quadratic formula, we can make some observations:

The $\pm$ will give x two different solutions, if the $\sqrt{b^2-4ac}$ term isn't zero.

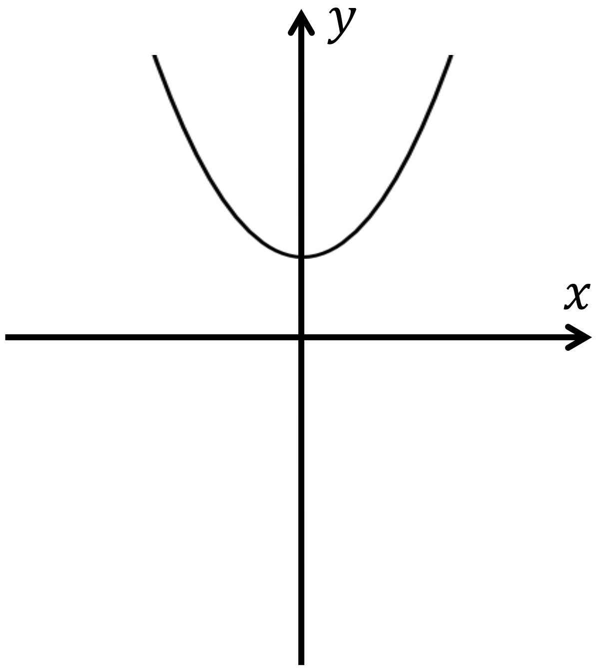
$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The $\boldsymbol{b^2 - 4ac}$ term can't be negative, as we can't have a negative number inside a square root.

- The $\boldsymbol{b^2 - 4ac}$ term is called the **discriminant** (and has the symbol Δ).
- These observations mean that:
  - If $\Delta < \mathbf{0}$, there will be **no solutions**.
  - If $\Delta = \mathbf{0}$, there will be **only one solution**.
  - If $\Delta > \mathbf{0}$, there will be **two solutions**.

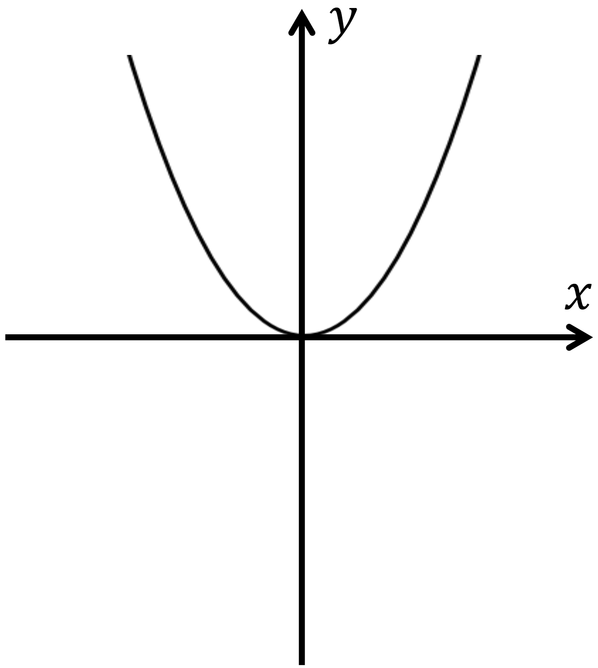
- Graphically:
- $\Delta < 0$

no solutions



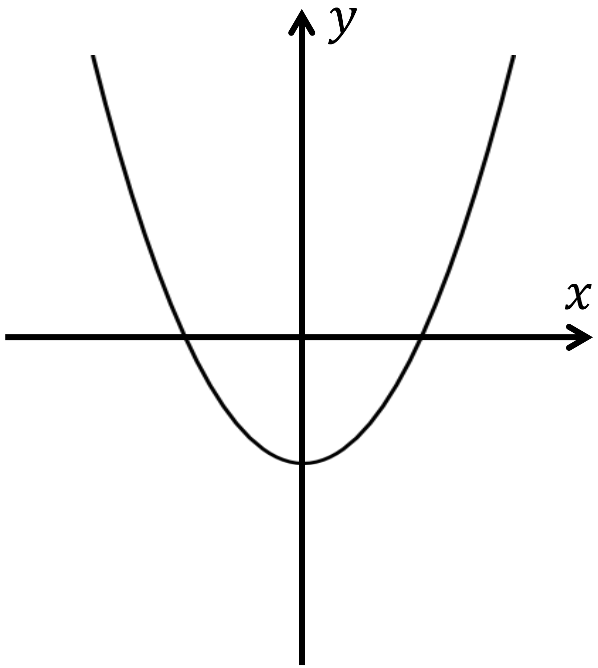
$\Delta = 0$

one solution



$\Delta > 0$

two solutions



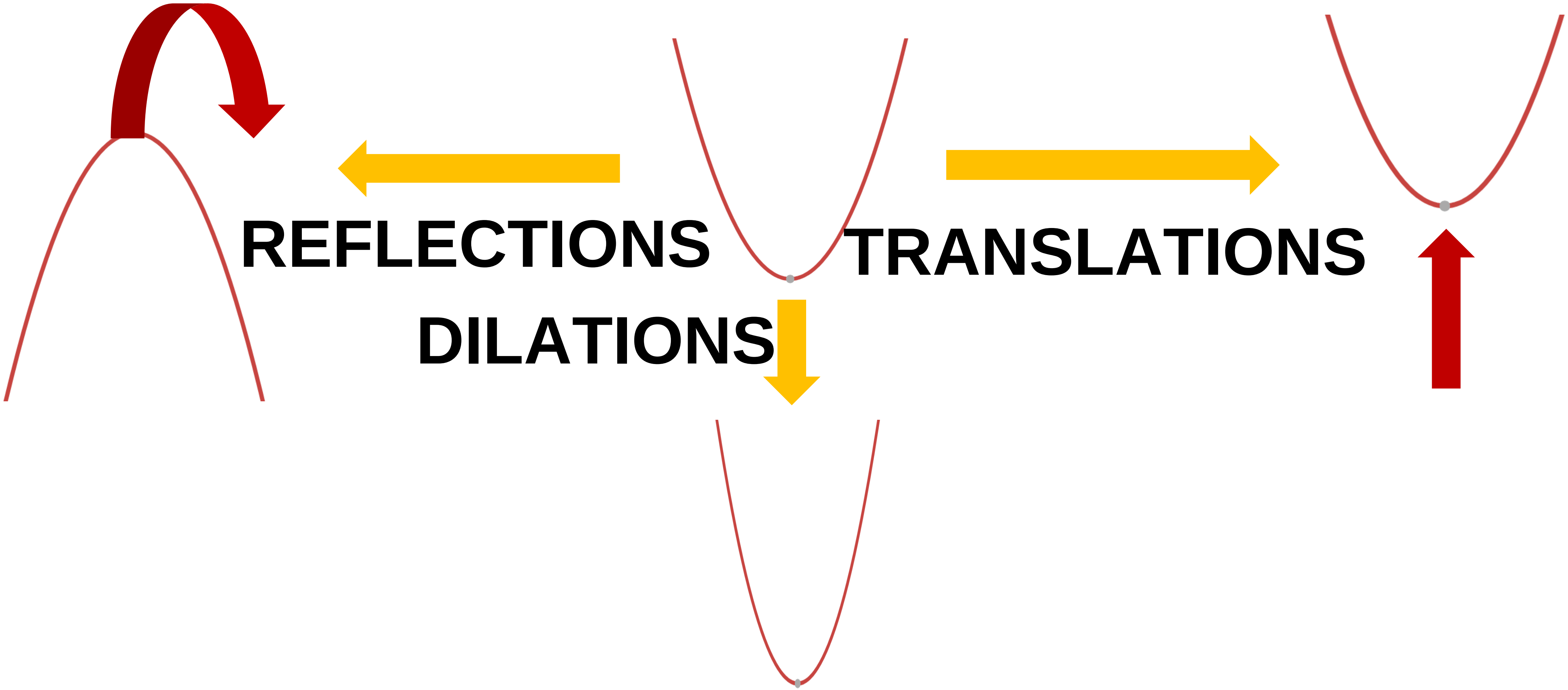
Let $f: R \to R, y = x^2 + mx + 4$, where m in a real constant.

a. Find the values of m for which the graph of $y = f(x)$ has exactly one x - intercept.

Q

TYPES OF TRANSFORMATIONS

- There are three types of transformations:



TYPES OF TRANSFORMATIONS

- Dilation by factor a from the y axis (alternatively, dilation by factor a in the direction of the x axis): affects x values
- Dilation by factor b from the x axis (alternatively, dilation by factor b in the direction of the y axis): affects y values

- Reflection in the y axis: affects x values
- Reflection in the x axis: affects y values

- Translation c in the positive/negative y direction (alternatively, translation c up/down): affects y values
- Translation d in the positive/negative x direction (alternatively, translation d right/left): affects x values

Why do we generally do translations after dilations and reflections when transforming?

A. because maths teachers are mean
B. translations are affected by the other transformations
C. only negative translations need to be done last
D. it's easier to visualise dilations if they are done first

TYPES OF TRANSFORMATIONS

- Unless ***specifically*** stated, transformations should be done in this order:

DILATIONS

REFLECTIONS

TRANSLATIONS

- Why is this so?

If dilations or reflections are done after translations, then the translations are altered

COORDINATE METHOD

- The coordinate method takes a little longer, but has far less room for errors so is a good method to use until you are confident
- Dilation a from y axis, b from x axis, reflection in x and y axes, translation c up and d right

$$(x, y) \rightarrow (ax, by) \rightarrow (-ax, -by) \rightarrow (\underline{-ax + d}, \underline{-by + c})$$

$$x' = -ax + d$$

$$\underline{y' = -by + c}$$

$$x = -\frac{1}{a}(x' - d) \qquad y = -\frac{1}{b}(y' - c)$$

$$x = -\frac{1}{a}(x' - d) \qquad y = -\frac{1}{b}(y' - c)$$

From here the above equations can be substituted into the function

$$-\frac{1}{b}(y' - c) = f\left(-\frac{1}{a}(x' - d)\right)$$

Q

A function $f(x) = \frac{1}{x}$ undergoes the following transformations

- Dilation 2 from y and 6 from x
- Reflection in the x axis
- Translation 1 up and $\frac{1}{2}$ right

State the transformed function g(x) in the form $-\frac{a}{bx+c} + d$ where a, b, c, and d are integers

REPLACEMENT METHOD

- The replacement method is faster but requires more memorisation, its derived from the replacement method

- Dilation a from y axis, b from x axis, reflection in x and y axes, translation c up and d right

$$b \times f\left(\frac{x}{a}\right)$$
$$-b \times f\left(-\frac{x}{a}\right)$$
$$-b \times f\left(-\frac{x-d}{a}\right) + c$$

REPLACEMENT METHOD

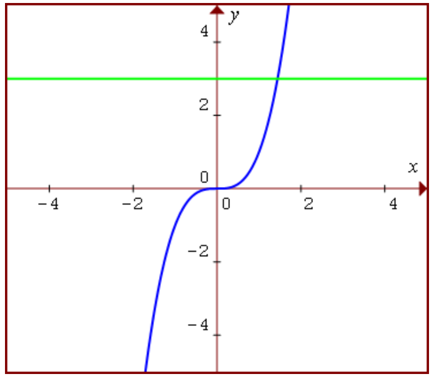
A function $f(x) = \frac{1}{x}$ undergoes the following transformations

- Dilation 2 from y and 6 from x
- Reflection in the x axis
- Translation 1 up and $\frac{1}{2}$ right

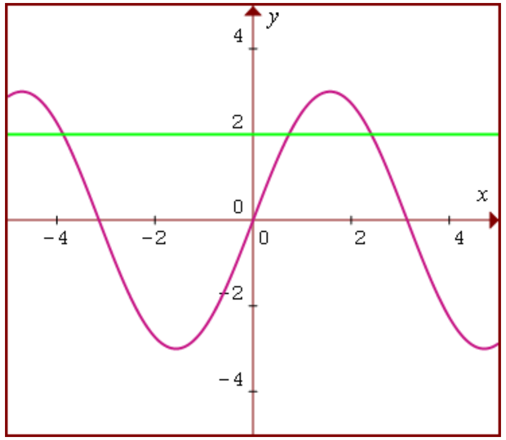
Using the replacement method, state the transformed function g(x) in the form $-\frac{a}{bx+c} + d$ where a, b, c, and d are integers

- A **one-to-one function** is a function where **no y-value repeats**.

- We can use the **horizontal line test** to determine if a function is one-to-one.
- To pass the test, the function should never touch the line twice (or more).



One-to-one



Not one-to-one

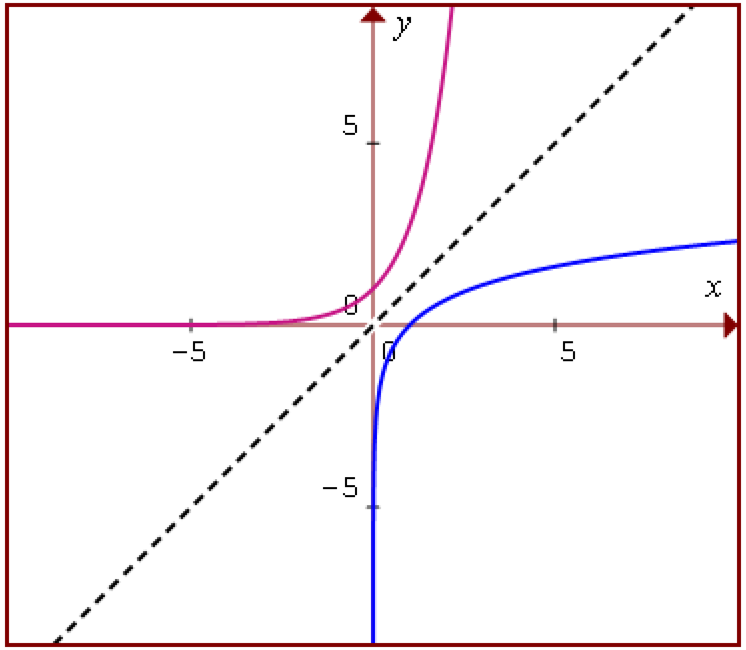
Inverse function

- Denoted f^{-1}, i.e. $y = f^{-1}(x)$
- Has the property $f^{-1}\big(f(x)\big) = x$

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & f(x) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

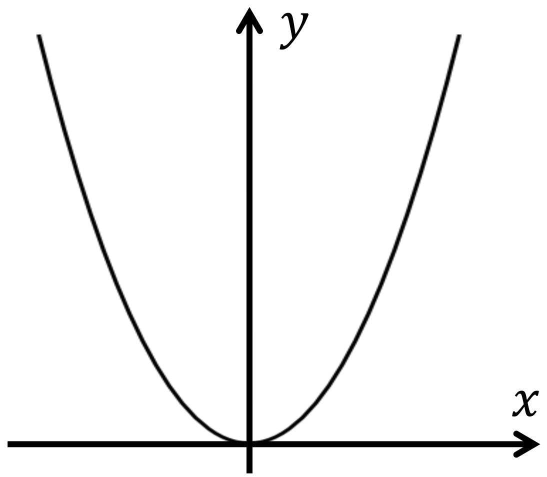
Geometrical meaning of the inverse

- The graphs of f and f^{-1} are related by **reflection** across $y = x$
- Note below both graphs are one-to-one

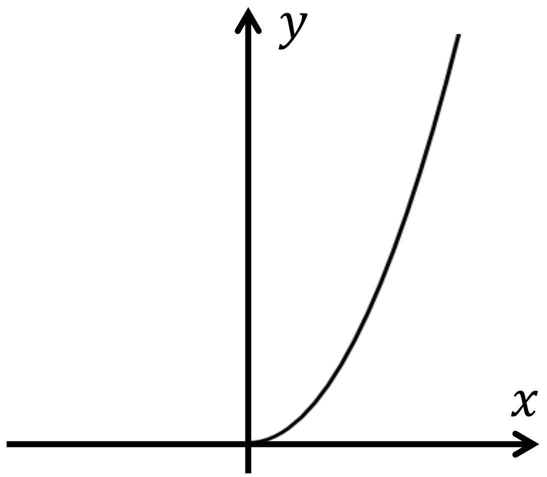


- If a function is not one-to-one, we can force it to be one-to-one.
- We can do this by **restricting the function's domain**.

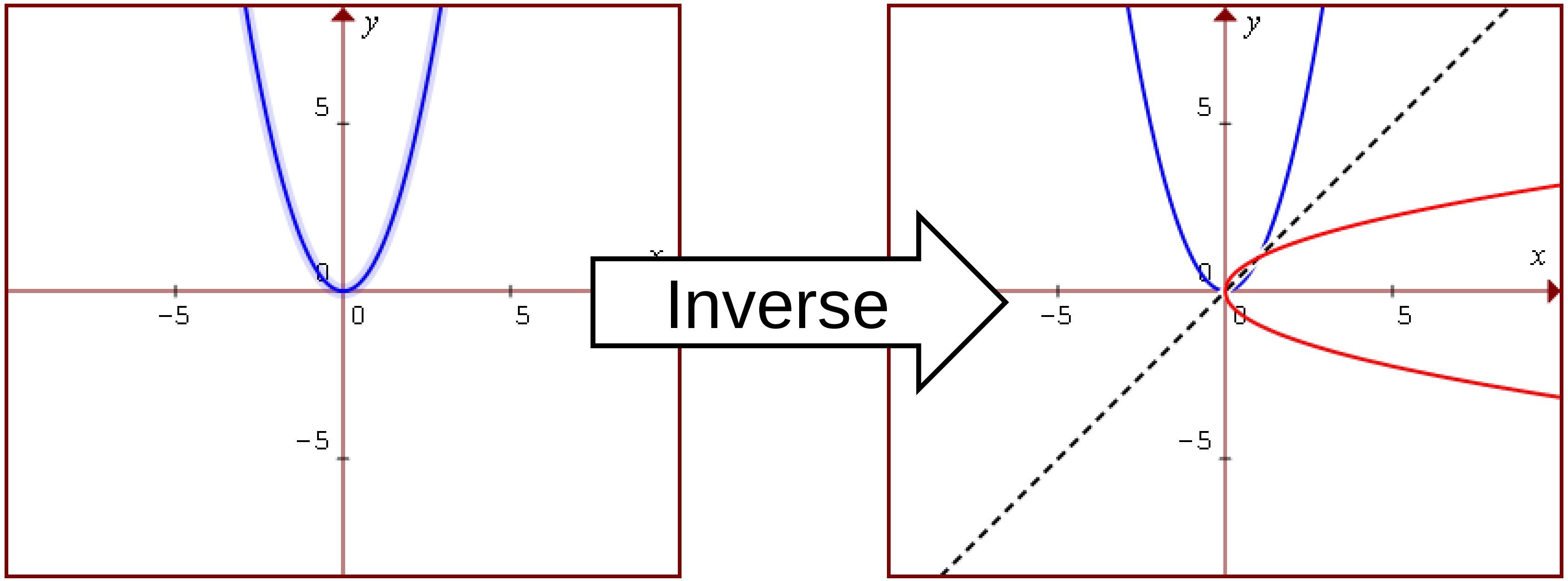
- For example, $f: R \to R$, $f(x) = x^2$ is not a one-to-one function.



- But if the domain is restricted to $[0, \infty)$, then it is one-to-one.



What if f is not one-to-one? We run into problems!



This is why the inverse only exists for one-to-one functions!

How to find the rule for an inverse function

- For a function f, the inverse function is notated f^{-1}.
- If you're asked to find f^{-1}, the inverse function of f:

1. Write "**Let $\boldsymbol{y = f(x)}$**"
2. Write "**To invert, swap $\boldsymbol{x}$ and $\boldsymbol{y}$**", or something equivalent
3. Rewrite the equation $y = f(x)$, with **$\boldsymbol{x}$ and $\boldsymbol{y}$ swapped**
4. Rearrange the equation to make y the subject
5. Write "$\boldsymbol{y = f^{-1}(x)} = \ldots$"

Q

Consider the function $g: (1, \infty) \to R$, $g(x) = \frac{4x-3}{x-1}$. Find the rule for g^{-1}, the inverse function of g.

Q

Consider the function $h: (1, \infty) \to R$, $h(x) = x^2 - 2x + 3$. Find the rule for h^{-1}, the inverse function of h.

PROBABILITY DEFINITIONS

- Consider a scenario: you're going to roll a fair six-sided die.

- The rolling of the die is the random **experiment**.
- The experiment has six different **outcomes**, one for each side of the die.

- The **sample space** is the set of all possible outcomes, so our sample space is $\{1,2,3,4,5,6\}$.
  - Note that curly brackets are used here to denote a set.

Q

A drawer contains three red socks and two blue socks. If two socks are taken from the drawer without replacement, what is the sample space?

PROBABILITY DEFINITIONS

- So an outcome is a possible result of a random experiment.
- An **event** is a group of possible outcomes.

- For example, if we’re rolling a six-sided die:
  - The possible outcomes are $\{1,2,3,4,5,6\}$.
  - Rolling an even number is an event, containing several outcomes
    - It has a sample space $\{2,4,6\}$

A fair six-sided will be rolled. Let A be the event in which an odd number is rolled and let B be the event in which a number less than 5 is rolled.

List the sets of both A and B.

SET NOTATION

- ∩ (intersection): $A \cap B$ is a list of outcomes that are in both A and B
- ∪ (union): $A \cup B$ is a list of outcomes that are in A or B
- ′ (complement): A' is a list of outcomes that are not in A (the opposite of A)
- ∅ (null set): a set that has nothing in it (it is empty)
- ε (epsilon): sometimes epsilon is used to list the whole sample space of a random experiment

If $A = \{1,3,5\}$, $B = \{1,2,3,4\}$ and $\varepsilon = \{1,2,3,4,5,6\}$, list the following sets:

a. $A \cap B$

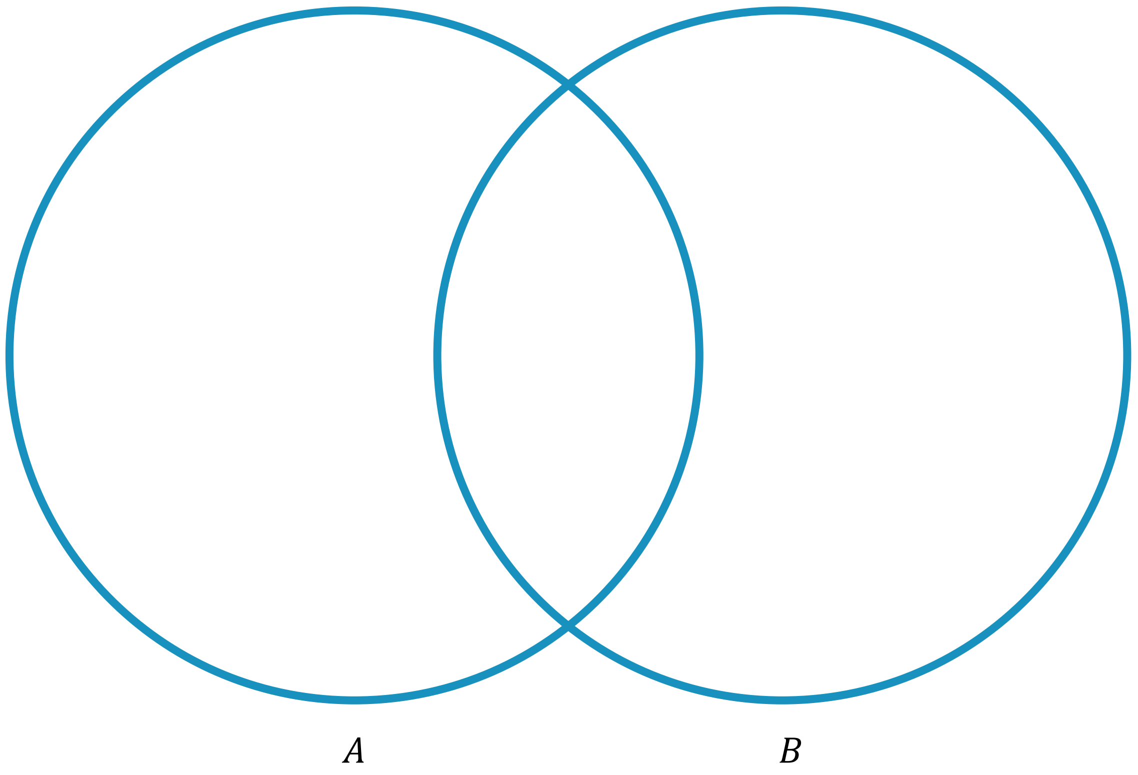
b. $A \cup B$

c. A'

d. $A' \cap B$

VENN DIAGRAMS

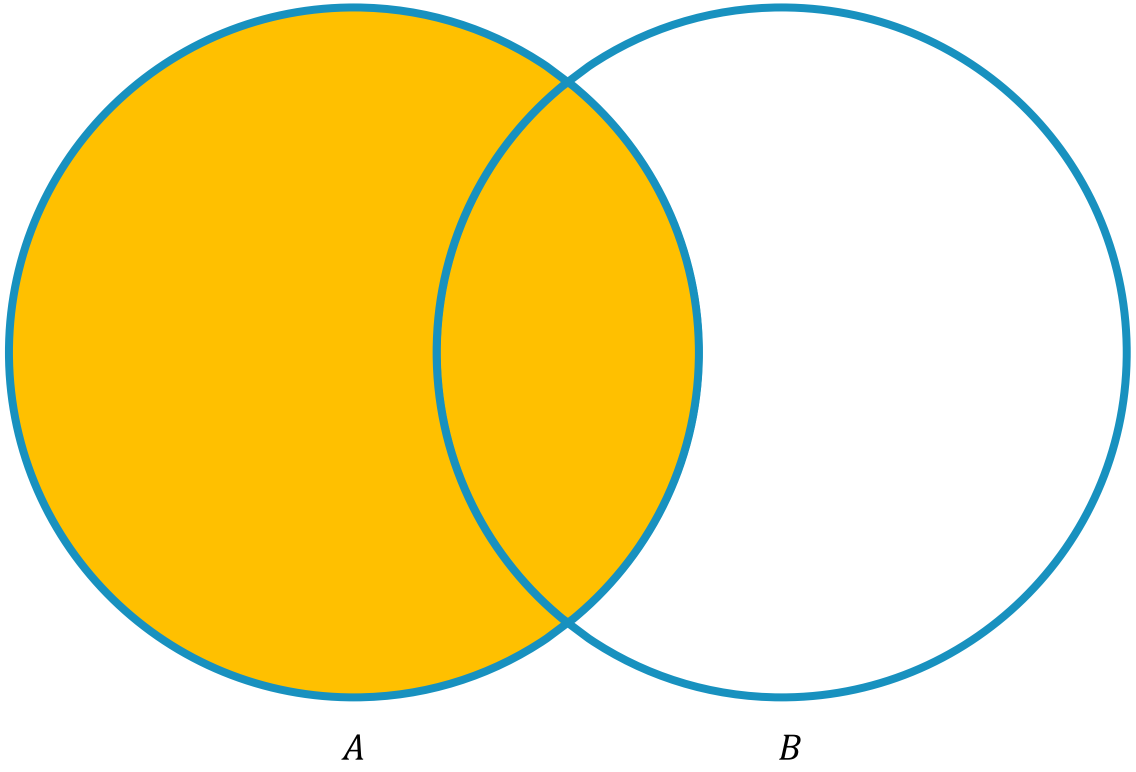
- One way to **visualise** a sample space is through **Venn diagrams**.



VENN DIAGRAMS

- One way to **visualise** a sample space is through Venn diagrams

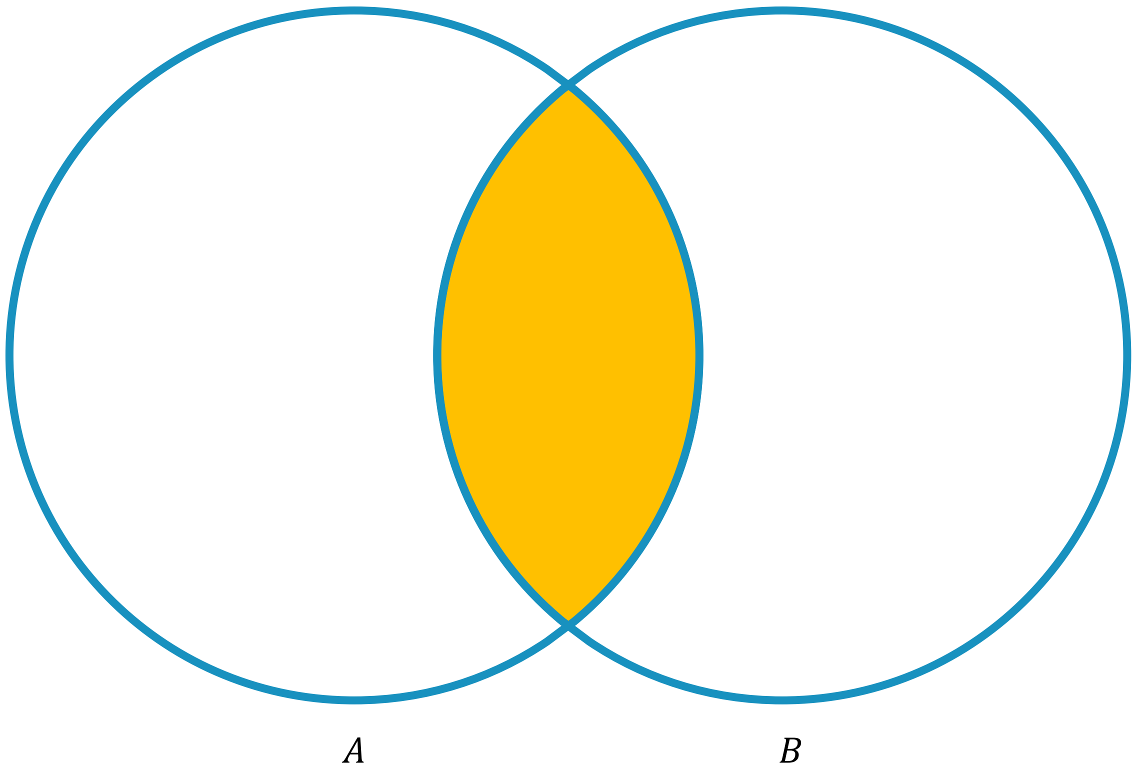
A



VENN DIAGRAMS

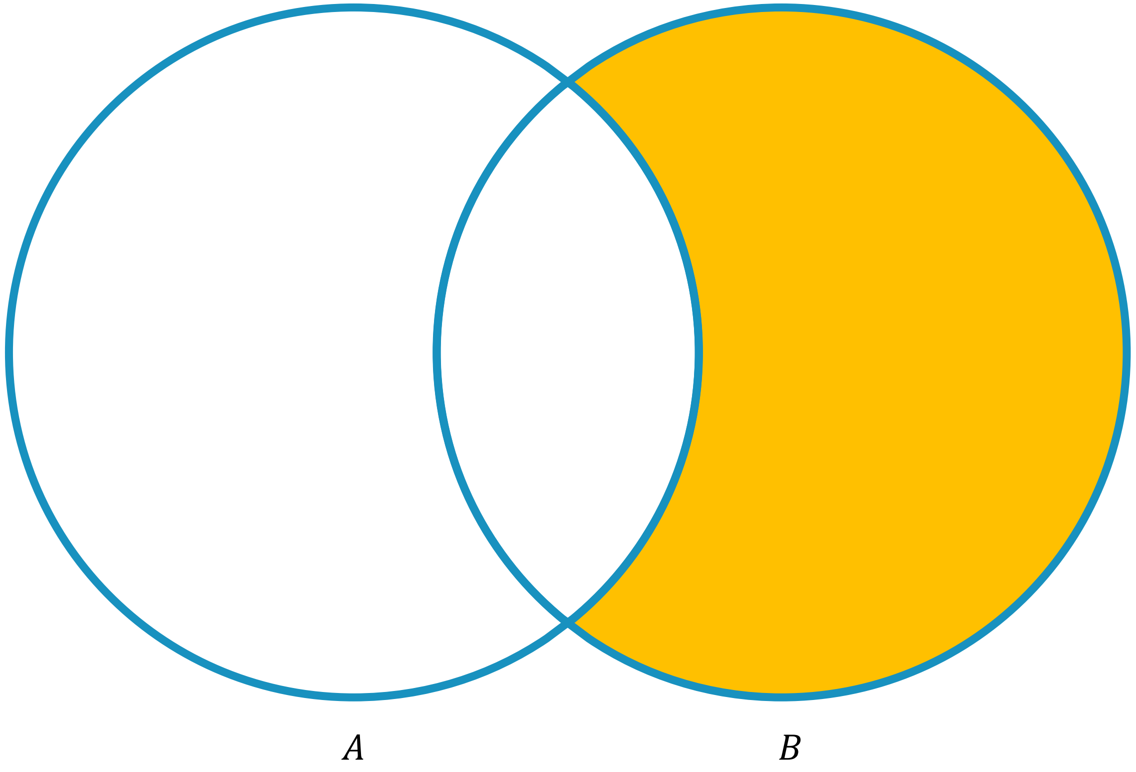
- One way to **visualise** a sample space is through Venn diagrams

$$A \cap B$$



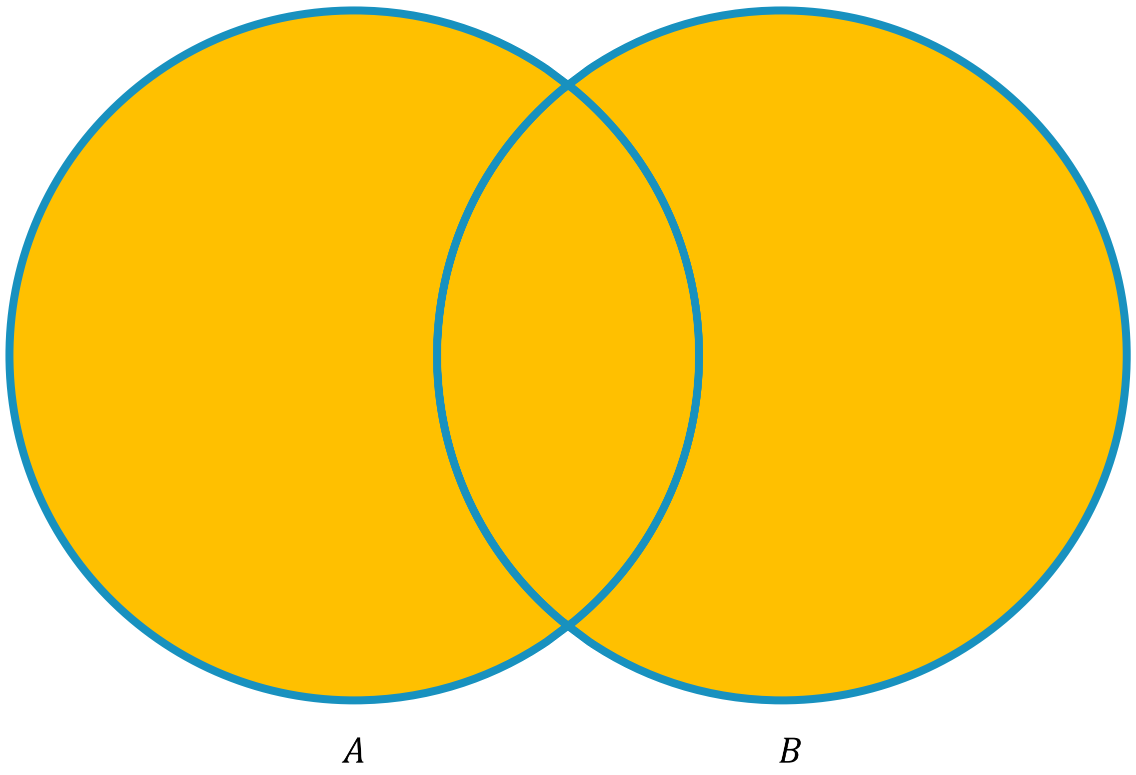
- One way to **visualise** a sample space is through Venn diagrams

$$A' \cap B$$

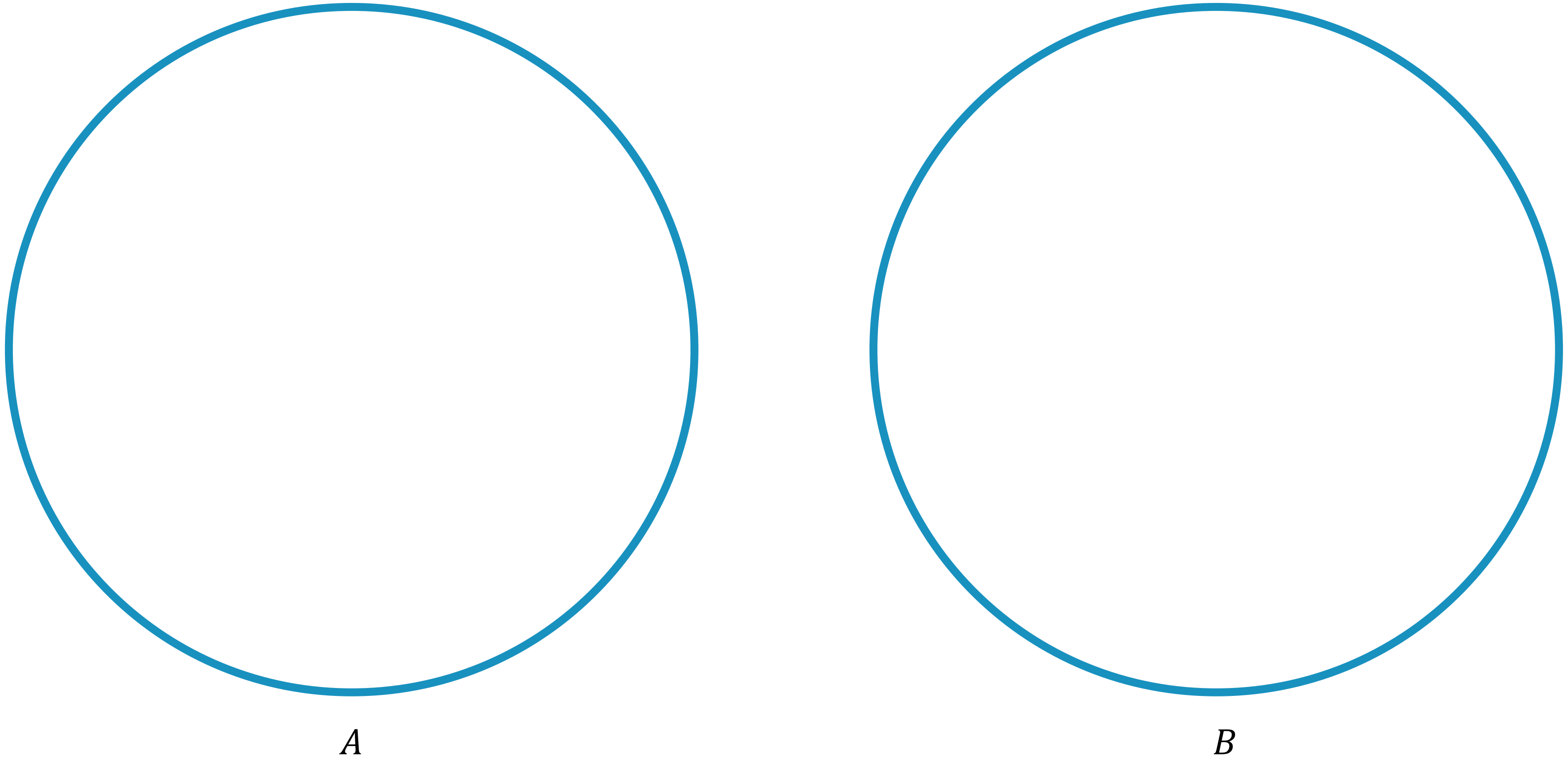


- One way to **visualise** a sample space is through Venn diagrams
- A handy formula is $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$$A \cup B$$

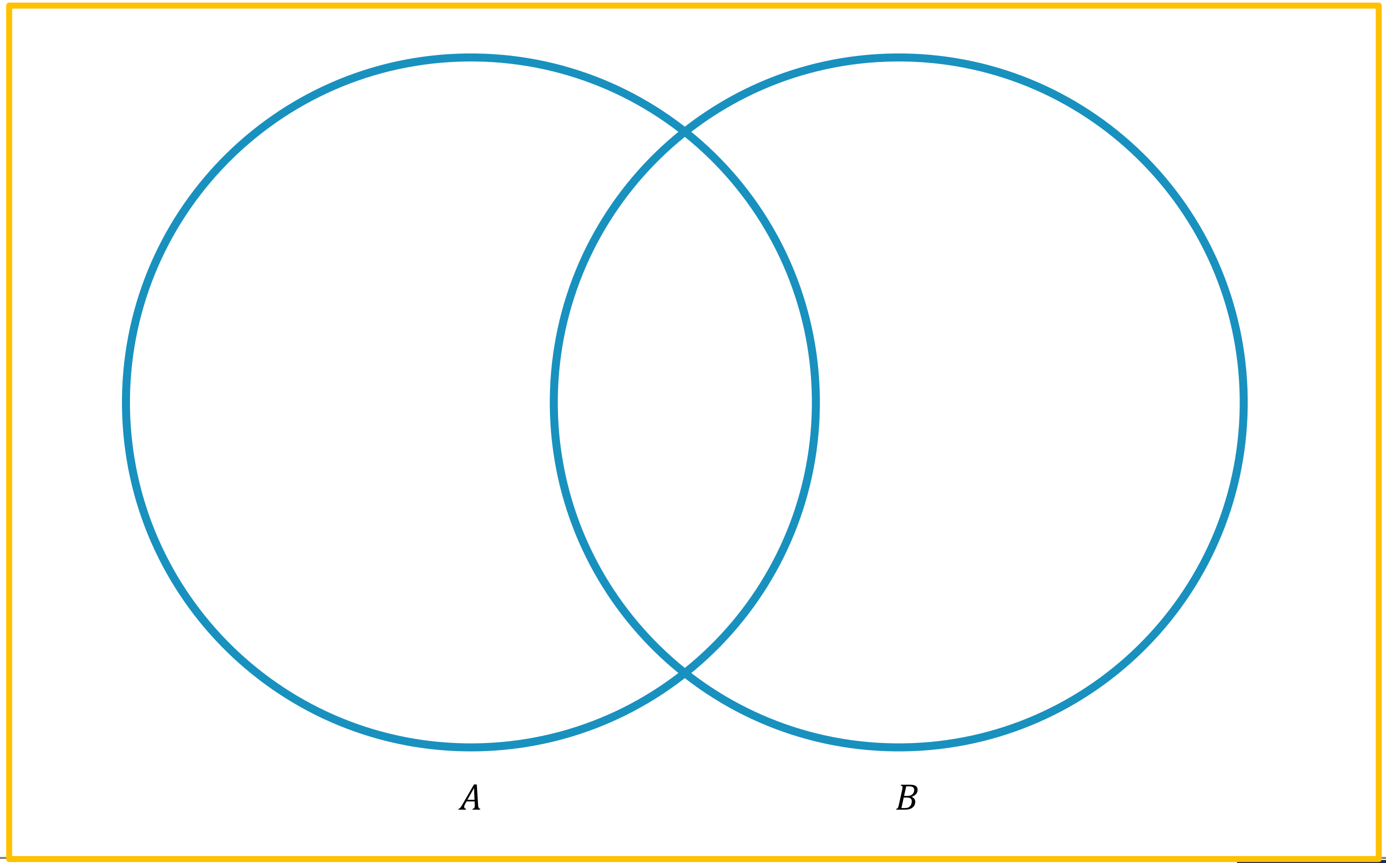


- If there is no overlap between A and B (so if $A \cap B = \emptyset$ or $\Pr(A \cap B) = 0$), then the events A and B are **mutually exclusive**.



VENN DIAGRAMS

- One way to **visualise** a sample space is through Venn diagrams
- Sometimes we put a box around the Venn diagrams to contain outcomes that are not part of either event



If $A = \{1,3,5\}$, $B = \{1,2,3,4\}$ and $\varepsilon = \{1,2,3,4,5,6\}$, represent the information in a Venn diagram.

The fundamental properties of probabilities

$$\Pr(\text{outcome}) = \frac{\#\text{ favourable outcomes}}{\#\text{ total outcomes}}$$

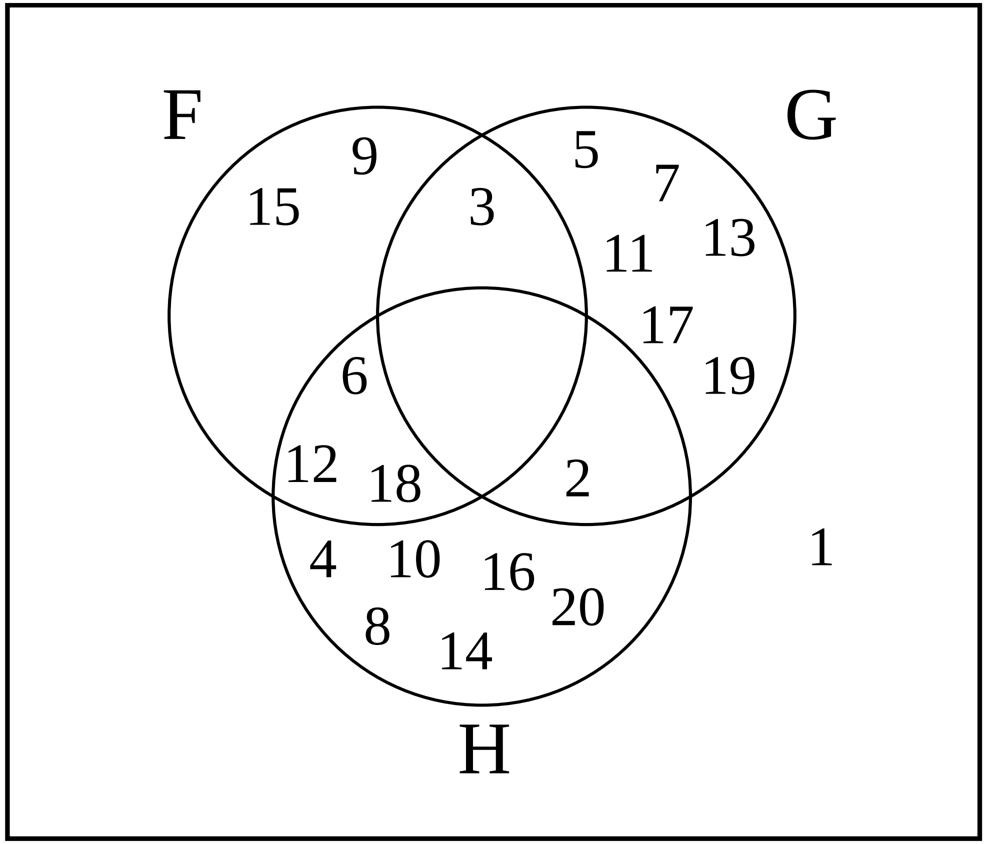
$$0 \leq \Pr(\text{outcome}) \leq 1$$

$$\sum_{\text{all outcomes}} \Pr(\text{outcome}) = 1$$

VENN DIAGRAMS

Using the same Venn Diagram as before, calculate:

$\Pr(F)$, $\Pr(G)$, and $\Pr(F \cap G)$



Without looking at the Venn Diagram, calculate:

$$\Pr(F \cup G)$$

CONDITIONAL PROBABILITY

- Knowing that one event has occurred affects the probability that another event has also occurred

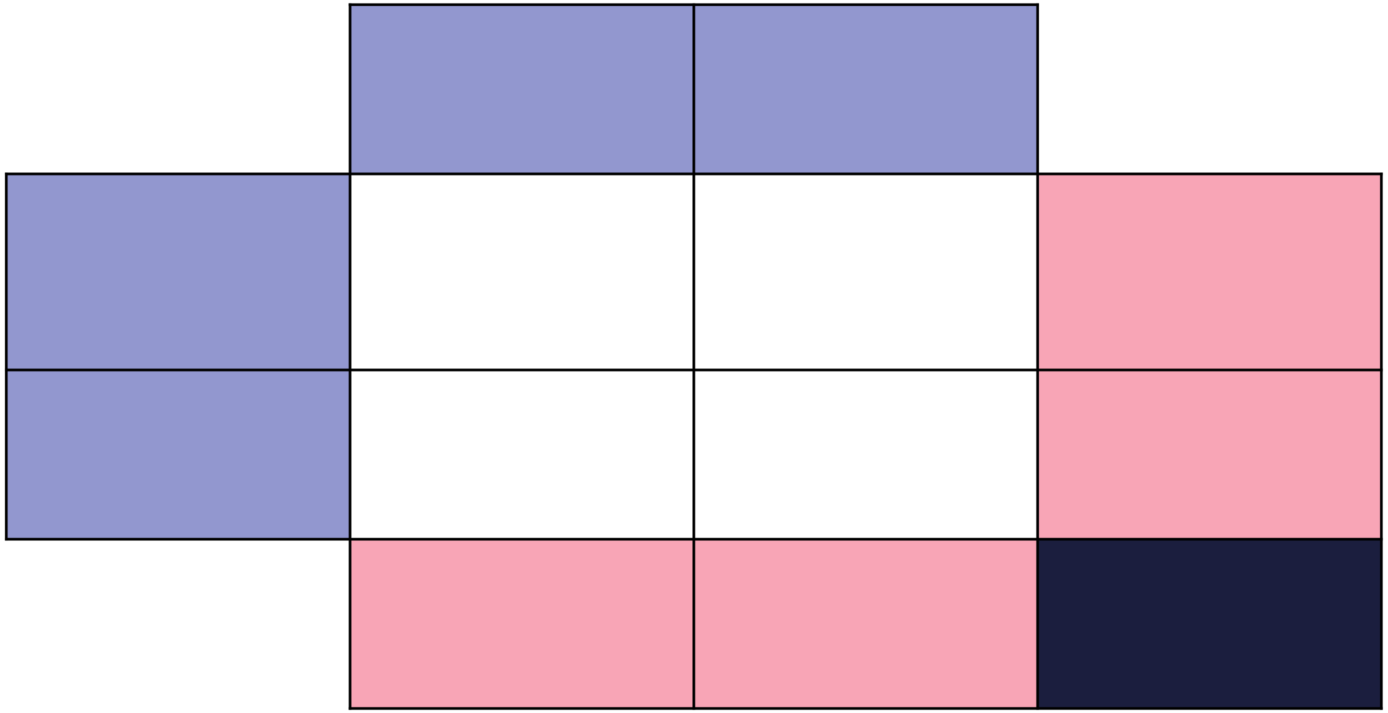
$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

- For example:
  - If it has rained the day before, there is usually a higher chance that it will rain the next day
  - If the card drawn from a deck is black, there is a higher chance that it is spades than before

- Another useful way of displaying probability information is through a **Karnaugh map** (probability table)

	B	B'	
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
	$\Pr(B)$	$\Pr(B')$	1

The probability that Cynthia walks to school and it is raining is 0.1. The probability that is doesn’t rain is 0.6 and the probability that Cynthia walks is 0.7. Using a Karnaugh map, what is the probability that on any given day she doesn’t walk and it isn’t raining?



The probability that Cynthia walks to school when it is raining is 0.1. The probability that is doesn’t rain is 0.6 and the probability that Cynthia walks is 0.7.

It is raining today, what is the probability that Cynthia walks?

	W	W'	
R	0.1	0.3	0.4
R'	0.6	0	0.6
	0.7	0.3	1

INDEPENDENT EVENTS

- Sometimes if one event occurs, it does not change the probability of another event occurring. These events are **independent**, and satisfy this formula:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

- For example, flipping a coin and whether it will rain tomorrow are independent events: whether the coin is heads or tails will not change the chance or rain.

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

Why is this so?

If A and B are independent, then B occurring will not affect the probability of A occurring

$$\mathbf{Pr(A|B) = Pr(A)}$$

So using the conditional probability formula:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A)$$

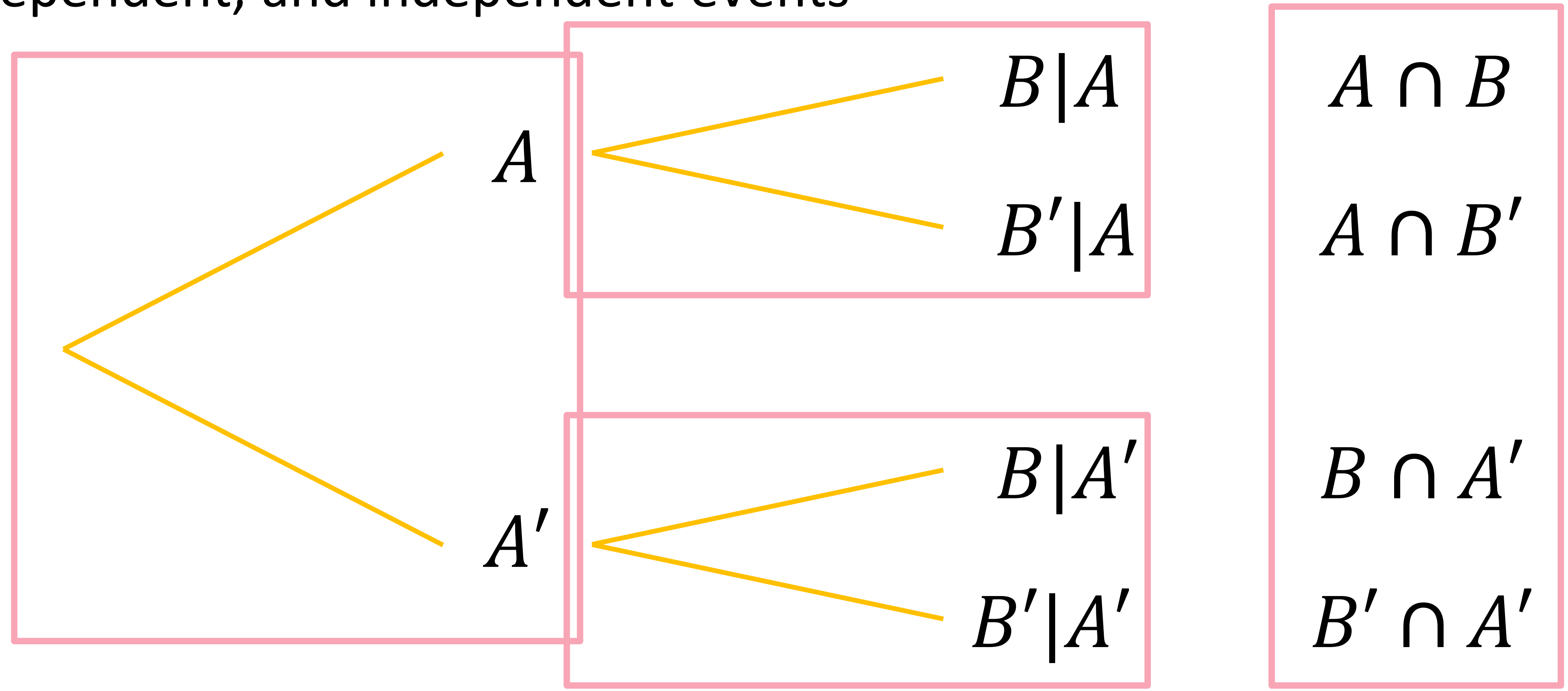
$$\Pr(A) \times \Pr(B) = \Pr(A \cap B)$$

The probability that Cynthia walks to school when it is raining is 0.1. The probability that is doesn’t rain is 0.6 and the probability that Cynthia walks is 0.7.

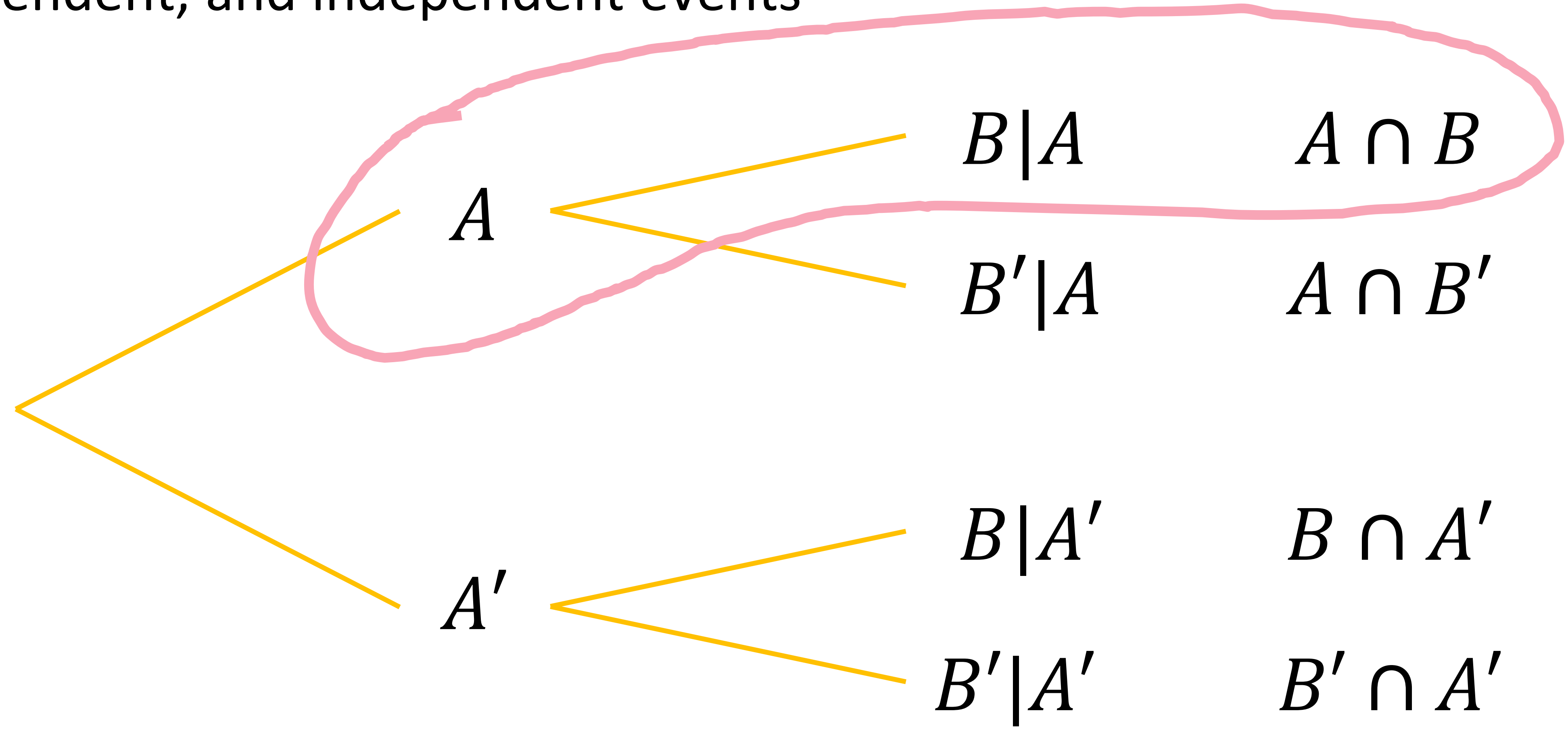
Are Cynthia walking and it raining independent events?

	W	W'	
R	0.1	0.3	0.4
R'	0.6	0	0.6
	0.7	0.3	1

- Tree diagrams can be useful to visualise probabilities for both dependent, and independent events



- Tree diagrams can be useful to visualise probabilities for both dependent, and independent events



TREE DIAGRAMS

- Tree diagrams can be useful to visualise probabilities for both dependent, and independent events

$$\Pr(A) \times \Pr(B|A) = \Pr(A \cap B)$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

A useful note:

If you have to say ‘I need this to happen **and** this to happen’, then we **multiply** their probabilities

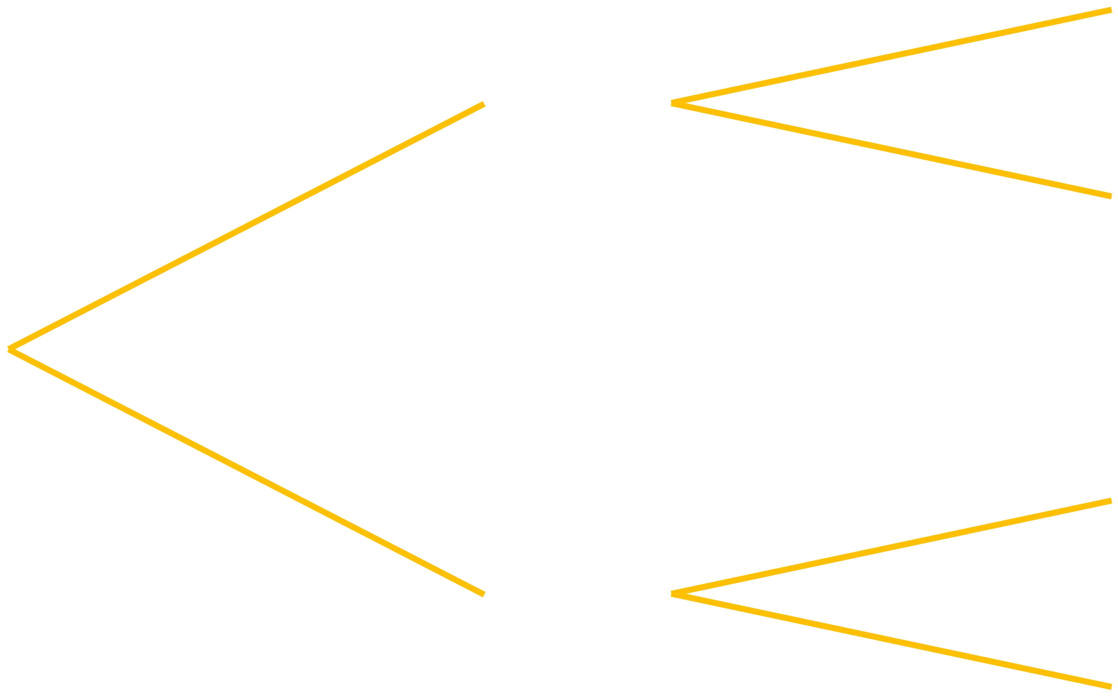
AND $\rightarrow \times$

If you have to say ‘I need this to happen **or** this to happen’, then we **add** their probabilities

OR $\rightarrow +$

TREE DIAGRAMS

In Blairsville, if it rains the previous day, the chance of rain today is 0.6. If it doesn't rain the previous day, the chance of rain today is 0.3. Using a tree diagram, if it rains on Monday, what is the probability that it rains on Wednesday?



COMBINATIONS

- If probability is

$$\Pr(\text{outcome}) = \frac{\#\text{ favourable outcomes}}{\#\text{ total outcomes}}$$

- It will be useful for us to have a fast way of calculating favourable outcomes

- This is where combinations come in

COMBINATIONS

- Combinations tell us the number of ways we can select r successes from a group of size n

$$\binom{n}{r} = {}^{n}C_{r}$$

$$= \frac{n!}{r!\,(n-r)!}$$

- For example: if you need to get 4 questions right out of 5 on a test, ${}^{5}C_{4}$ will tell us how many different ways you can pass

FACTORIAL

- An exclamation mark denotes a factorial

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

- $0! = 1$ (see notes if you would like to know why)

- Something handy to remember when cancelling is

$$n! = n \times (n-1) \times (n-2)!$$

$$5! = 5 \times 4 \times 3!$$

If a coin is flipped 7 times, how many different outcomes are there where 3 heads are flipped?

That’s all folks!

Key skills

- Math is super cool, remember that!
- For more fun, check out TuteSmart, I’ll be there teaching away.

Reminders

- Thank you all so much for coming!
- Last minute questions in the chat?
- Hopefully it helped – and remember to take it slow and build that strong base for 3/4